Announcements

1) Today is the last day to drop individual classes

2) HW #4 up on Canvas, due Tuesday next week (Webwork + written)

Recall: (local maxima)

For $z = f(x,y)$, (a,b) is an absolute maximum if there exists $r > 0$ such that

$$f(a,b) > f(x,y) \text{ when}$$

$$\|(a,b) - (x,y)\| < r.$$

Local Minima

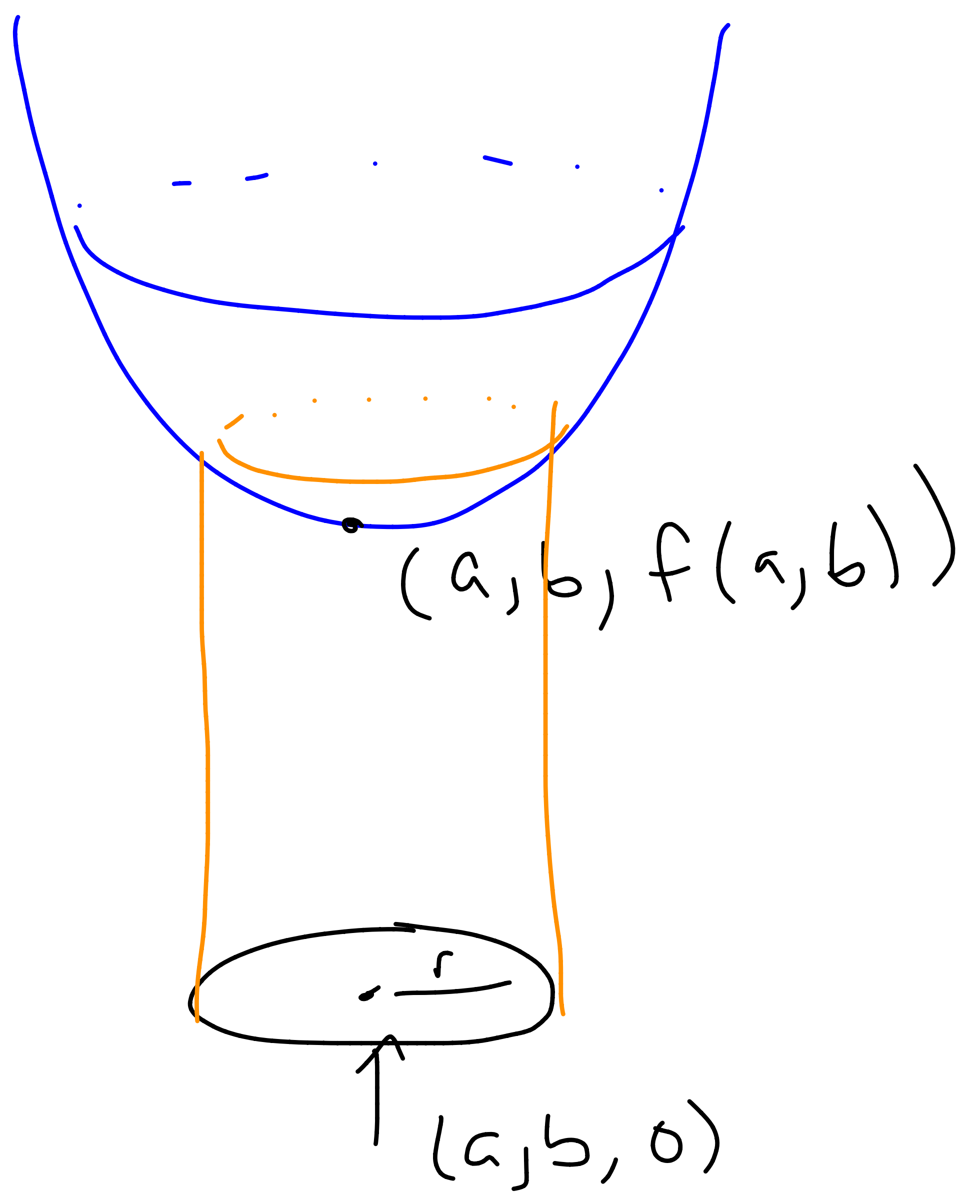
Same kind of definition,

except

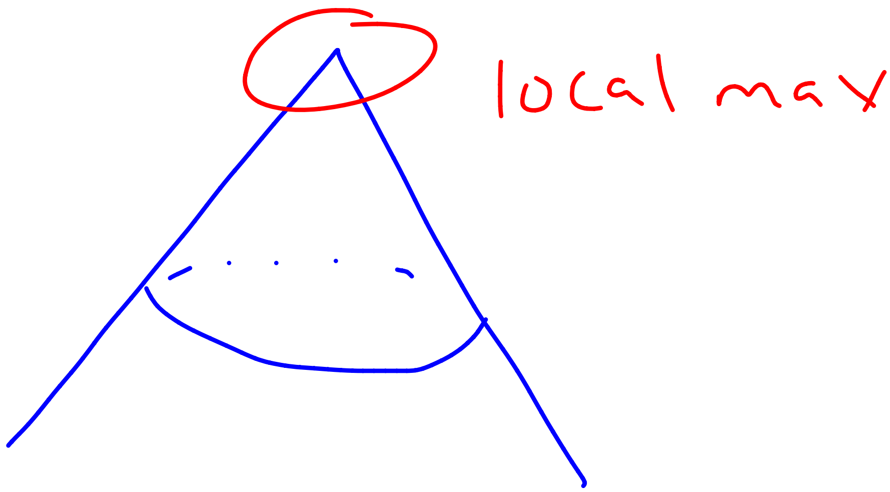
$$\boxed{f(a,b) < f(x,y)}$$

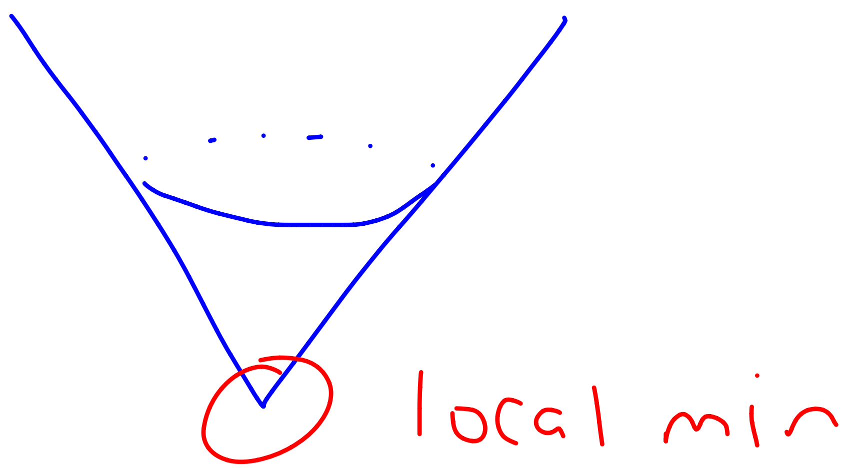
when $\|(a,b)-(x,y)\| < r$

Picture



Non-differentiable local maxima and minima





Calc I Consequence

Suppose $z = f(x,y)$ has a local maximum or minimum at (a,b). If $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ both exist at (a,b), then

$$\boxed{\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0}$$

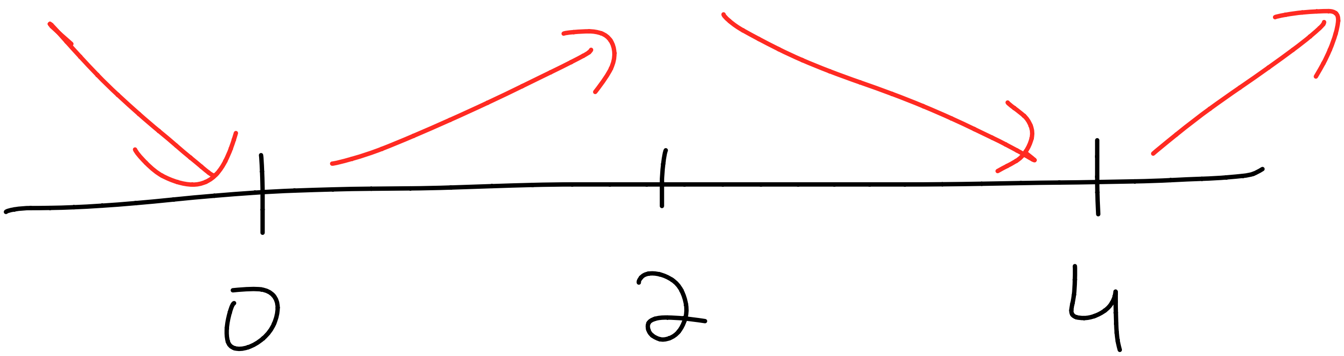
Critical Points

A critical point for $z = f(x,y)$ is any point (a,b) where either f is not differentiable at (a,b) or

$$\boxed{\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0}$$

when (a,b) is in the domain of f.

Recall: 1^{st} Derivative test in Calc I – find all critical points of f. Draw them on a number line. Plug points in between critical points into f':



Foolproof!

There is no 1^{st} Derivative Test Now!

Second Derivative Test:

Find critical points where f' exists (f' is zero).

Take the second derivative, plug these points in.

1) $f'' > 0$, local min

2) $f'' < 0$, local max

3) $f'' = 0$ test fails!

Second Derivative Test

(2 variables)

Let $z = f(x,y)$ be a function of two variables, and suppose (a,b) is a critical point.

Suppose $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist, so

$$\frac{\partial f}{\partial x}(a,b) = \frac{\partial f}{\partial y}(a,b) = 0.$$

Then $\longrightarrow$

Let

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial y \partial x}\frac{\partial^2 f}{\partial x \partial y}$$

$$= \det \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

Then

1) If $D(a,b) > 0$ and

$\frac{\partial^2 f}{\partial x^2}(a,b) < 0$, then

(a,b) gives a local max

2) If $D(a,b) > 0$ and

$\frac{\partial^2 f}{\partial x^2}(a,b) > 0$, then

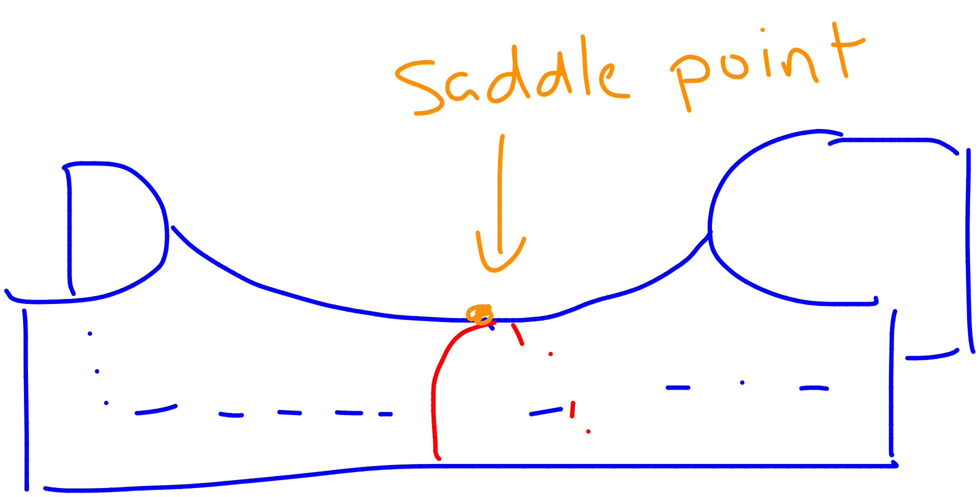
(a,b) gives a local min

3) If $D(a,b) < 0$, then

f has neither a local max

nor a local min, but

a saddle point



4) If either $D(a,b)=0$

or $\dfrac{\partial^2 f}{\partial x^2}(a,b)=0$ in a) or b)

the test fails and

you know nothing!

Example 1: $f(x,y) = xy(1-x-y)$

Find all critical points, classify as local max, local min, or saddle points.

$$f(x,y) = xy - x^2y - xy^2$$

Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = y - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = x - x^2 - 2yx$$

For a critical point,

$$0 = y - 2xy - y^2$$
$$= x - x^2 - 2xy$$

$$0 = y - 2xy - y^2$$ solve for x

$$2xy = y - y^2$$

$$2xy = y - y^2$$

if $y \neq 0$, divide both sides by y:

$$2x = 1 - y$$

$$x = \frac{1-y}{2}.$$

Plug into $\frac{\partial f}{\partial y} = 0$

$$x - x^2 - 2xy = 0$$

$$\left(\frac{1-y}{2}\right) - \left(\frac{1-y}{2}\right)^2 - 2\left(\frac{1-y}{2}\right)y = 0$$

Pull out $\frac{1-y}{2}$, divide

if $\frac{1-y}{2} \neq 0$

$$1 - \left(\frac{1-y}{2}\right) - 2y = 0$$

$$2 - (1-y) - 4y = 0$$

$$1 = 3y$$

$\boxed{y = 1/3}$, $\boxed{x = \frac{1-y}{2} = \frac{2/3}{2} = 1/3}$

If $\boxed{y=0}$,

$\frac{\partial f}{\partial x} = 0$ gives nothing.

$\frac{\partial f}{\partial y} = 0$ gives

$$x - x^2 = 0$$

$$\boxed{x = 0, 1}$$

If $\boxed{x=0,}$

$\frac{\partial f}{\partial x}=0$ gives

$y-y^2=0,$

$\boxed{y=0,1}$

Critical Points

$$(0,0),\ (0,1),\ (1,0),\ (1/3, 1/3)$$

Compute second-order partials

$$\frac{\partial f}{\partial x} = y - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = x - 2xy - x^2$$

$$\frac{\partial^2 f}{\partial x^2} = -2y, \quad \frac{\partial^2 f}{\partial y^2} = -2x$$

Clairault

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}} = 1 - 2x - 2y$$

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= 4xy - (1-2x-2y)^2$$

$$\frac{\partial^2 f}{\partial x^2} = -2y$$

@ $(0,0)$ $\quad D(0,0) = -1$

Saddle point

@ $(1,0)$ $\quad D(1,0) = -1$

Saddle point

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= 4xy - (1-2x-2y)^2$$

$$\frac{\partial^2 f}{\partial x^2} = -2y$$

@ $(0,1)$ $\quad D(0,1) = -1$

saddle point

@ $(1/3, 1/3)$ $\quad D(1/3, 1/3)$

$$= 4/9 - 1/9 = 1/3 > 0$$

$$\frac{\partial^2 f}{\partial x^2}(1/3, 1/3) = -2/3 < 0$$

local max

Absolute maxima and Minima

Recall: Calc I case: A continuous function on a closed interval attains its maximum and minimum on that interval.

Analog in $\mathbb{R}^2$

Replace "closed interval"

with "closed and

bounded set":

A subset S of $\mathbb{R}^2$ is

bounded if there is an

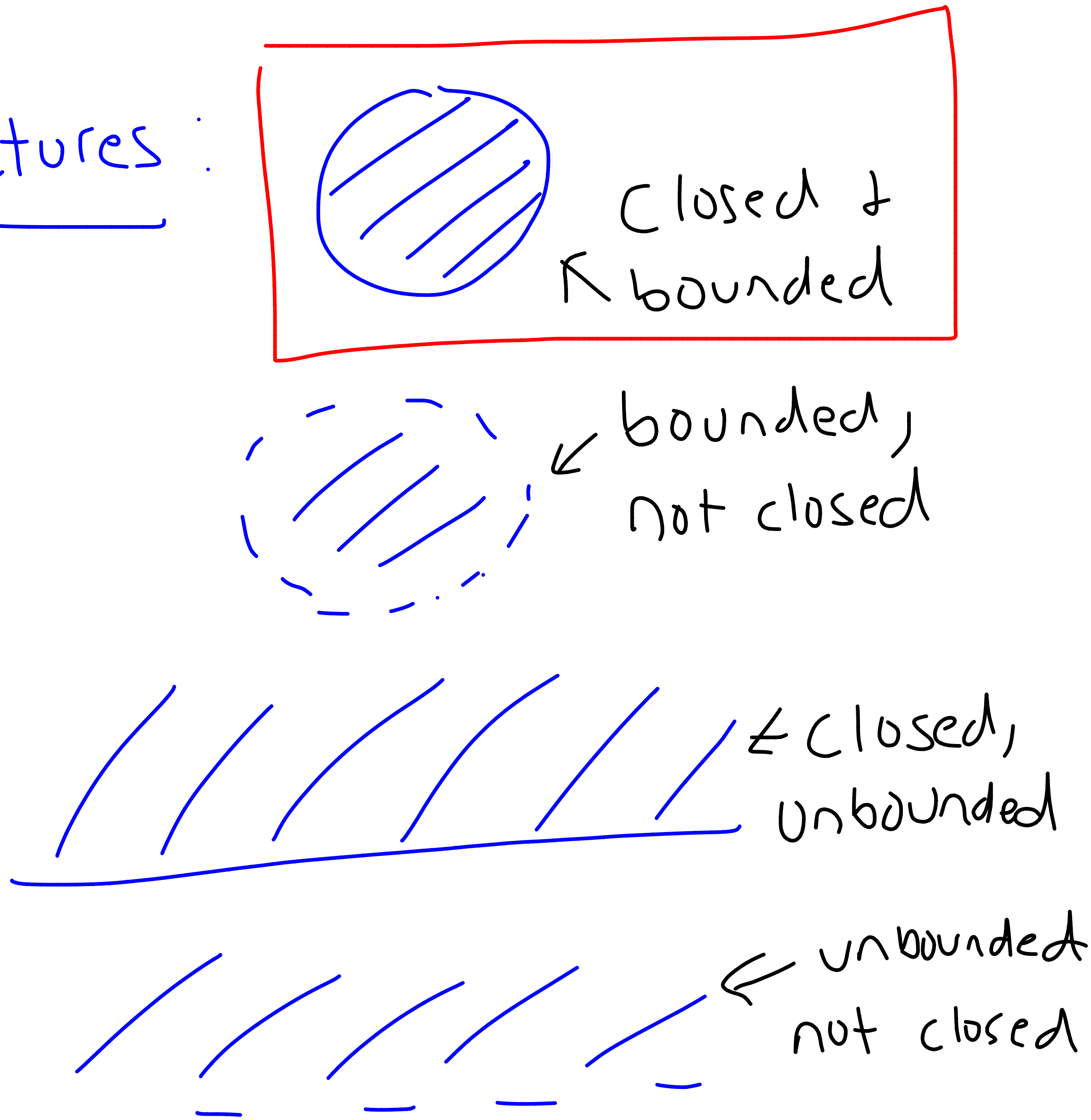
$r > 0$ such that

$\|(x,y)\| < r$ when

(x,y) is in S.

A subset S of $\mathbb{R}^2$ is closed if it contains its "boundary"

Pictures:



Procedure for Finding Absolute Maxima and Minima

Works for continuous functions on closed and bounded sets

1) Find all critical points NOT on the boundary

2) Plug the boundary equation into the function - this gives a function of one variable. Find all critical points of this function

3) Plug all points found in 1) and 2) into your original function. Biggest value = max, smallest = min.

Example 2:

$f(x,y) = x^2y^4$,

find absolute maximum and minimum on the unit disk:

$$\{(x,y) \mid x^2+y^2 \leq 1\}$$

Boundary: unit circle

$$x^2+y^2=1.$$

1) Find partials.

$$\frac{\partial f}{\partial x} = 2xy^4, \quad \frac{\partial f}{\partial y} = 4x^2y^3$$

$(0, y)$ or $(x, 0)$

all give

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} = 0.$$

2) Next class!